Resurgence, exact WKB and quantum geometry and the sign problem

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based on:
1501.05671, 1308.1108,16xx.xxxx with G.Dunne, M. Ünsal
and
1510.03258, 1512.08764, 1604.00956, $(16\text{xx.xxxx})^2$,
with A. Alexandru, P. Bedaque, G.Ridgway, N. Warrington

Many expansions in physics are divergent-asymptotic:

$$f(\hbar) \sim \sum_{n=0}^{\infty} c_n \hbar^n \quad , \quad c_n \sim n!$$

QM, QFT, strings, hydrodynamics, fluid/gravity...

some examples: *(beware! highly incomplete list)*

- quartic/cubic oscillator, Mathieu, Zeeman, Stark, ...
- Dyson instability, weak field Euler-Heisenberg, QFT in dS/AdS background, large N, ...
- genus expansion in string theory ($c_g \sim (2g)!$) [Shenker]
- boost invariant conformal hydrodynamics [Heller,Spalinski; GB, Dunne]

How can we assign a physical value to an asymptotic series?

- *Mathieu equation* [GB, Dunne; 1501.05671]

$$-\frac{\hbar^2}{2}\frac{d^2\psi}{dz^2} + cos(z)\,\psi = u(N,\hbar)\,\psi$$

- Relevant for SUSY gauge theories in D=2,4 [Nekrasov, Shatashvili]
  quantum integrable systems $\Leftrightarrow$ SUSY gauge theories
- Encodes the vacua of $\mathcal{N}=2$, $SU(2)$ theory in its spectrum
  $u \Leftrightarrow \mathrm{tr}\langle\Phi^2\rangle$, moduli space coord.
- ODE $\Leftrightarrow$ 2D integrable models [Dorey, Tateo; Voros; Bazhanov, Fateev, Lukyanov, Zamolodchikov; ...]
- Related to conformal block expansion via null vector decoupling equation [Kashani-Poor, Troost; Piatek, Pietrykowski]
- Wilson loops in $\mathcal{N}=4$ (via AdS/CFT and Pohlmeyer Reduction) [Kruczenski et. al]

Trans-series

near $u \sim -1$, tightly bound states, tunneling exponentially suppressed

$$\begin{aligned} u(N,\hbar) \quad \sim \quad & -1 + \hbar\left[N+\frac{1}{2}\right] - \frac{\hbar^2}{16}\left[\left(N+\frac{1}{2}\right)^2+\frac{1}{4}\right] \\ & -\frac{\hbar^3}{16^2}\left[\left(N+\frac{1}{2}\right)^3+\frac{3}{4}\left(N+\frac{1}{2}\right)\right] - \dots \\ & + \underbrace{e^{-\frac{S_I}{\hbar}}\sum_n \hbar^n f_n(N)\cos\theta}_{1-instanton} + \underbrace{e^{-\frac{2S_I}{\hbar}}\sum_n \hbar^n g_n(N,\theta)}_{2-instanton} + \dots \end{aligned}$$

trans-monomials:

$\hbar^n$ (perturbative fluctuations), $e^{\frac{-k\,S_I}{\hbar}}$ (multi instantons), $\log(-1/\hbar)^l$ (quasi zero modes)

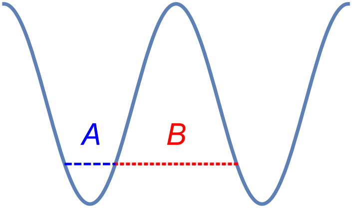
Resurgence relations

- In addition to the large order - low order relations between perturbative and non-perturbative expansions, there is a surprising low order - low order relation between them!
- allows one to *fully construct* the non-perturbative fluctuations from perturbative data.
- valid everywhere in the spectrum

Trans-series and WKB expansion

$$\psi \sim e^{\frac{i}{\hbar}Q(z,u;\hbar)} \;\Rightarrow\; Q'^{\,2} + i\hbar Q'' - 2(u - V(z)) = 0 \quad \text{(Ricatti eqn.)}$$

- Formal expansion: $Q \sim \sum_{n=0}^{\infty} \hbar^n Q_n(z,u)$
- Two cycles: A and B



WKB actions: [Dunham]

$$a(u;\hbar) = \frac{1}{2\pi}\int_A Q' dz \sim \sum_{n=0}^{\infty} a_n(u)\hbar^{2n}$$

$$a^D(u;\hbar) = \frac{1}{2\pi}\int_B Q' dz \sim \sum_{n=0}^{\infty} a_n^D(u)\hbar^{2n}$$

- perturbative : $a(u;\hbar) = \frac{\hbar}{2}(N + 1/2) \Rightarrow u_{p.t.}(N)$
- non-perturbative (tunneling): $\Delta u = \frac{2}{\pi}\frac{\partial u}{\partial N} e^{-\frac{2\pi}{\hbar}\mathcal{I}m[a^D]}$

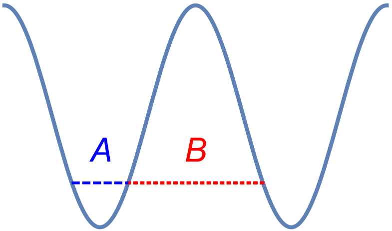
Geometry and WKB

- Set $\hbar = 0$ for now.
- Classically the (complex) phase space can be identified with the moduli space of complex tori.
- $u \Leftrightarrow$ moduli space parameter

$$u = \frac{p^2}{2} + \cos z \quad \Rightarrow \quad x \equiv \cos z, \quad y = \frac{\dot{x}}{\sqrt{2}}$$

$$y^2 = (x^2 - 1)(x - u)$$ genus-1 elliptic curve

Geometry and WKB



WKB actions: integrals of abelian differentials over the two independent cycles of torus

$$a_0(u) = \frac{\sqrt{2}}{2\pi} \int_A \sqrt{u - V(z)}\, dz = \frac{\sqrt{2}}{\pi} \int_A \frac{u - x}{y} dx$$

$$a_0^D(u) = \frac{\sqrt{2}}{2\pi} \int_B \sqrt{u - V(z)}\, dz = \frac{\sqrt{2}}{\pi} \int_B \frac{u - x}{y} dx$$

Geometry and WKB

a_0 and a_0^D are related via *Riemann bilinear identity*

$$a_0 \frac{da_0^D}{du} - a_0^D \frac{da_0}{du} = \frac{i}{2} \frac{S_I}{T}$$

$T = 2\pi$ =period of the harm. oscll. at the bottom of the well

- a_0, a_0^D: satisfy a Picard-Fuchs equation

$$4(1-u^2) a_0''(u) - a_0(u) = 0$$

- Bilinear identity $\Leftrightarrow$ Wronskian
- alternatively: $a_0^D(u) = \tau(u)\, a_0(u) - i \frac{S_{inst}}{\omega_0(u)}$

  where $\omega_0 = a_0'$, modular parameter: $\tau = \omega_0^D / \omega_0$

Geometry and WKB: Quantum corrections

$$a(u;\hbar) \sim \sum_{n=0}^{\infty} a_n(u)\hbar^{2n} \quad , \quad a^D(u;\hbar) \sim \sum_{n=0}^{\infty} a_n^D(u)\hbar^{2n}$$

All higher order actions are encoded in the lowest order (classical) action

$$a_n(u) = p_n(u)a_0(u) + q_n(u)a_0'(u) \ , \ a_n^D(u) = p_n(u)a_0^D(u) + q_n(u)a_0^{D\prime}(u)$$

- p_n, q_n: rational functions that follow from Schrödinger eqn.

Geometry and WKB: Quantum corrections

"quantum corrections" to the bilinear identity

[GB, Dunne]

$$\left(a - \hbar\frac{\partial a}{\partial \hbar}\right)\frac{\partial a^D}{\partial u} - \left(a^D - \hbar\frac{\partial a^D}{\partial \hbar}\right)\frac{\partial a}{\partial u} = \frac{2i}{\pi}$$

- connects the perturbative expansion to non-perturbative fluctuations order by order.
- valid *everywhere* in the spectrum [see talk by Dunne]
- SUSY inspired proof via Matone's relation [Gorsky, Milekhin]

$$\mathrm{P} = \mathrm{NP}$$

perturbative expansion:

$$u^{pt.}(N,\hbar) \;\sim\; -1+\hbar\left[N+\frac{1}{2}\right]-\frac{\hbar^2}{16}\left[\left(N+\frac{1}{2}\right)^2+\frac{1}{4}\right]+\ldots$$

$$\Updownarrow$$

band width (non-perturbative, 1-instanton+fluctuations) :

$$\Delta u_{1\,inst.}(N,\hbar) = \frac{\partial u^{pt.}}{\partial N}\; e^{S_I\int_0^{\hbar}\frac{d\hbar'}{\hbar'^{\,3}}\left(\frac{\partial u^{pt.}(N,\hbar')}{\partial N}-\hbar'+\frac{\hbar'^{\,2}(N+\frac{1}{2})^2}{S_I}\right)}$$

checked up to 3 loops via explicit calculation [Escobar-Ruiz, Shuryak, Turbiner]

Bigger picture

- perturbative $\Leftrightarrow$ non-perturbative connection exists for *any* potential whose spectral curve is genus-1
- The explicit relation is technically more complicated in the most general case (Picard-Fuchs equation for the action is 3^{rd} order).
- There is an infinite class of potentials (double well, triple well, quadruple well, ...) where it is simple (i.e. PF is 2^{nd} order)

  [GB, Dunne, Ünsal; 1605.xxxx]
- higher genus?

Bonus:

Beyond semi classics:
Attacking the sign problem with holomorphic gradient flow

with A. Alexandru, P. Bedaque, G.Ridgway, N. Warrington

1510.03258, 1512.08764, 1604.00956, $(16xx.xxxx)^2$

[Related work (Lefschetz thimbles): Christoferetti et. al., Fuiji et.al.]

Monte-Carlo method and the sign problem

a generic method to study strongly coupled systems

$$\langle \mathcal{O} \rangle = \frac{1}{Z} \int D\phi \, e^{-S[\phi]} \mathcal{O}[\phi] \quad \Rightarrow \quad \langle \mathcal{O} \rangle = \frac{1}{\mathcal{N}} \sum_a \mathcal{O}[\phi^{(a)}]$$

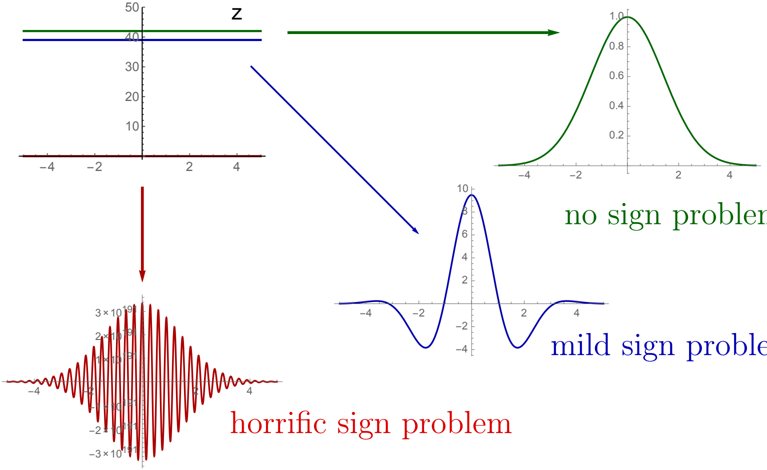
$\phi^{(a)}$ sampled according to the distribution $P[\phi] = e^{-S[\phi]}/Z$

what if S is complex ? as in: many-body systems with non-zero density, real time dynamics, QCD with non-zero θ [see talk by Cohen], ...

- one can try $\langle \mathcal{O} \rangle = \langle \mathcal{O}\, e^{-iS_I} \rangle_{S_R} / \langle e^{-iS_I} \rangle_{S_R}$: *reweighting*
- S_I grows with the volume $\rightarrow$ ~~reweighting~~

Idea: complexify the fields

example: $$\int_{-\infty}^{\infty} e^{-\frac{1}{4}(x+42i)^2} = 2\sqrt{\pi}$$



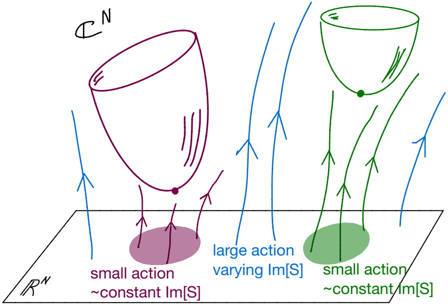
no sign problem

mild sign problem

horrific sign problem

How to do this in higher dimensions?

holomorphic gradient flow:



$$\frac{d\phi}{d\tau} = \overline{\frac{\partial S}{\partial \phi}}$$
$$\frac{d\phi_R}{d\tau} = \frac{\partial S_R}{\partial x} = \frac{\partial S_I}{\partial y}$$
$$\frac{d\phi_I}{d\tau} = \frac{\partial S_R}{\partial y} = -\frac{\partial S_I}{\partial x}$$

- family of manifolds with milder sign problem
- no runaways ! (as opposed to complex Langevin)
- Metropolis on these manifolds: contraction algorithm
  [Alexandru, GB, Bedaque, Ridgway, Warrington]
- sign problem $\Leftrightarrow$ potential barriers (multimodal distributions)

Relativistic Bose gas in 4d

complex scalar field: $\phi = \phi^1 + i\phi^2$

$$\mathcal{L} = |\partial_\mu \phi|^2 + (m^2 - \mu^2)|\phi|^2 + \mu\left(\phi^* \partial_0 \phi - \phi \partial_0 \phi^*\right) + \lambda|\phi|^4 + h(\phi^1 + \phi^2)$$

sign problem here!

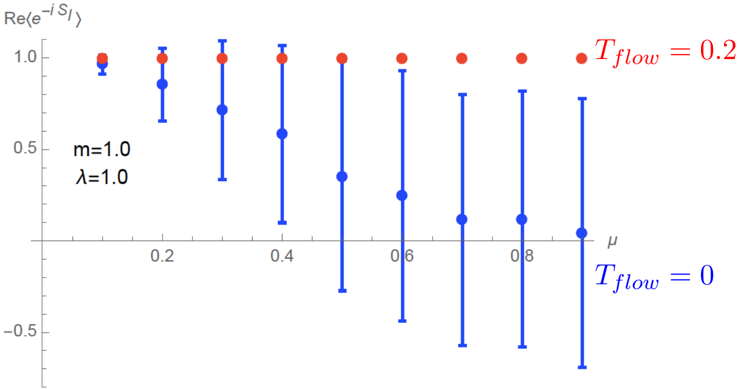
discretization:

$$\begin{aligned} S \;&=\; \sum_x \Big[\Big(4 + \frac{m^2}{2}\Big)\phi_x^a\phi_x^a + \frac{\lambda}{4}(\phi_x^a\phi_x^a)^2 - h(\phi_x^1 + \phi_x^2) \\ &\quad - \sum_{\nu=1}^{3} \phi_x^a\phi_{x+\hat{\nu}}^a - \cosh\mu\;\phi_x^a\phi_{x+\hat{0}}^a - i\sinh\mu\;\epsilon_{ab}\phi_x^a\phi_{x+\hat{0}}^b\Big] \end{aligned}$$

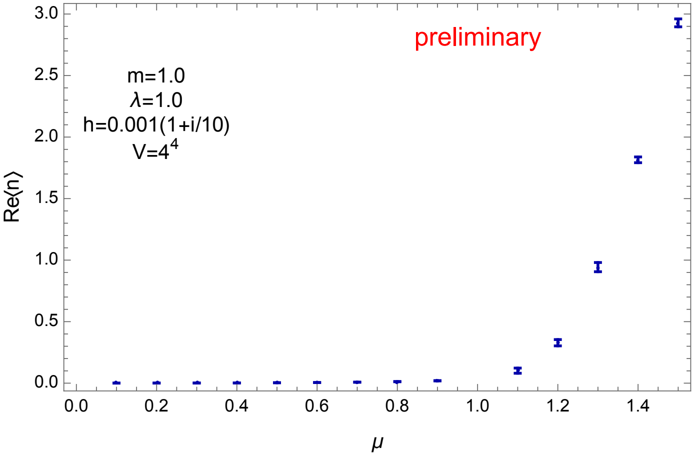
sign problem here!

Relativistic Bose gas in 4d

sign problem ?



Relativistic Bose gas in 4d

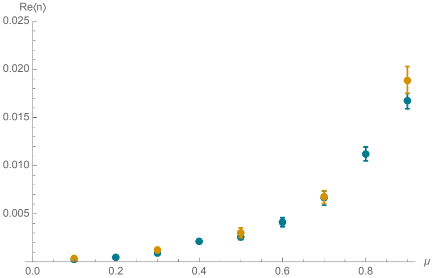


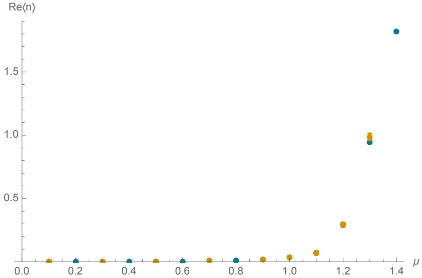
(deformation: $\mathbb{R}^N \to \mathcal{T}$)

Relativistic Bose gas in 4d

comparison with other computations

[Fuiji et.al., JHEP, 10:147, 2013 (Hybrid Monte-Carlo, yellow points)]





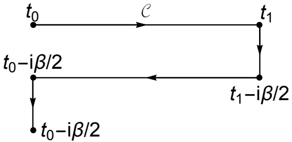
Real time physics from the lattice

Motivation: compute out-of-equilibrium correlators, transport coefficients etc.. non-perturbatively from first principles

main object:

$$\begin{aligned}\langle \mathcal{O}_1(t)\mathcal{O}_2(0)\rangle &= Tr[\mathcal{O}_1(t)\mathcal{O}_2(0)\, e^{-\beta H}] \\ &= Tr[e^{-iHt}\mathcal{O}_1(0)e^{iHt}\mathcal{O}_2(0)e^{-\beta H}]\end{aligned}$$

path integral representation: closed time contour [Schwinger, Keldsyh; ...]



$t_0 \sim t_0 - i\beta$

$$S_{SK}[\phi] = \int_{\mathcal{C}} dt\, L[\phi]$$

$$\langle \mathcal{O}_1(t)\mathcal{O}_2(t')\rangle = \frac{1}{Z}\int \mathcal{D}\phi\, \underbrace{e^{iS_{SK}[\phi]}}_{\text{terrible sign problem!!}}\, \mathcal{O}_1(t)\mathcal{O}_2(t')$$

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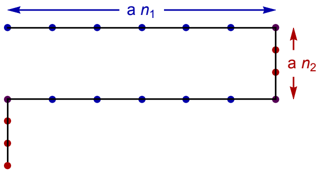


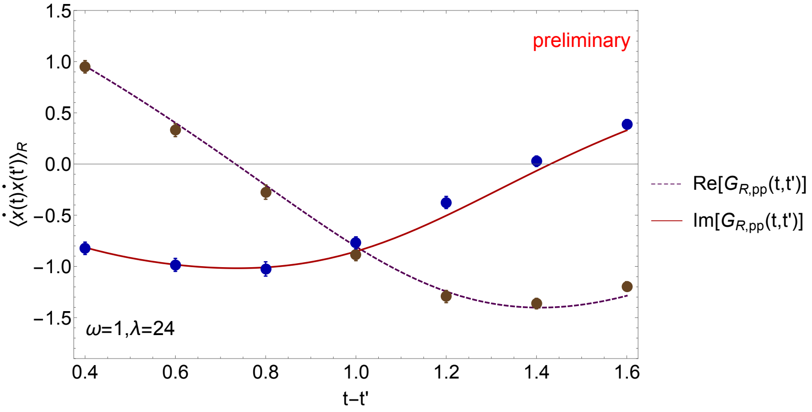
$t_0 \sim t_0 - i\beta$

$$S = -i\sum_{i=0}^{N}\Delta t_i\Big[\frac{1}{2}\left(\frac{x_{i+1}-x_i}{\Delta t_i}\right)^2 - \frac{V(x_{i+1})+V(x_{i-1})}{2}\Big]$$

$$\langle \mathcal{O}\rangle = \frac{\int dx_i\, e^{-S[x]}\,\mathcal{O}[x]}{\int dx_i\, e^{-S[x]}}$$

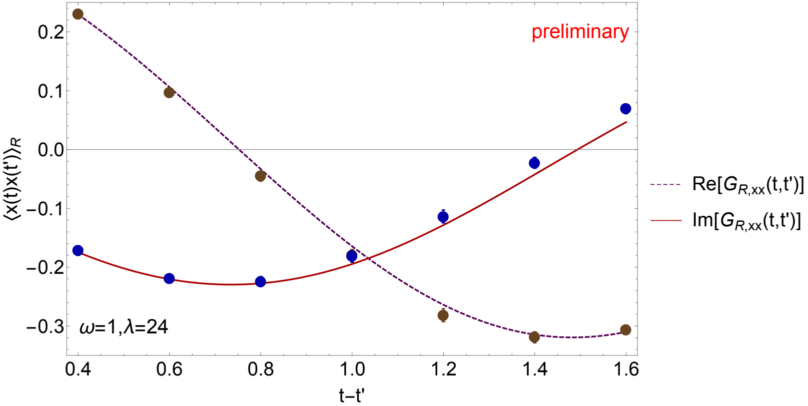
Real time physics, results: anharmonic oscillator

- consider $G(t,t') = \langle \dot{x}(t)\dot{x}(t') \rangle$
- response to an external force, analogue of conductivity



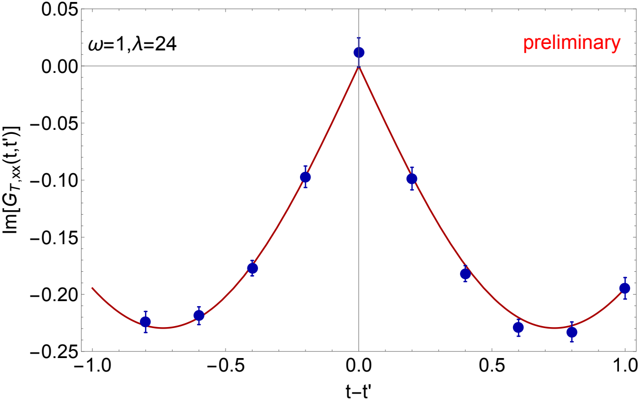
Real time physics, results: anharmonic oscillator

$\langle x(t)x(t')\rangle$ retarded



Real time physics, results: anharmonic oscillator

$\langle x(t)x(t')\rangle$ time-ordered



Real time physics: anharmonic oscillator

A remark:

This problem was also studied via complex Langevin

[Berges, Stamatescu,'05; Berges, Borsanyi, Sexty, Stamatescu, '06]

which converges to the wrong result for $T_{max} > \beta$.

Our approach does not have such a problem.

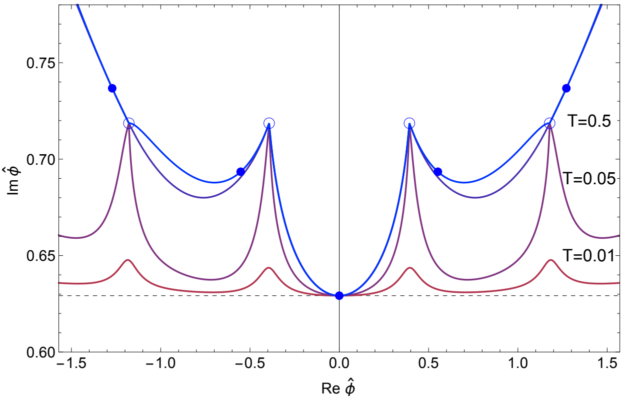
Conclusions

resurgence:

- geometry $\leftrightarrow$ WKB expansion: surprising P- NP connection.
- how general?, implications for gauge theories/ CFTs?, (topological) string theory?, higher genus?

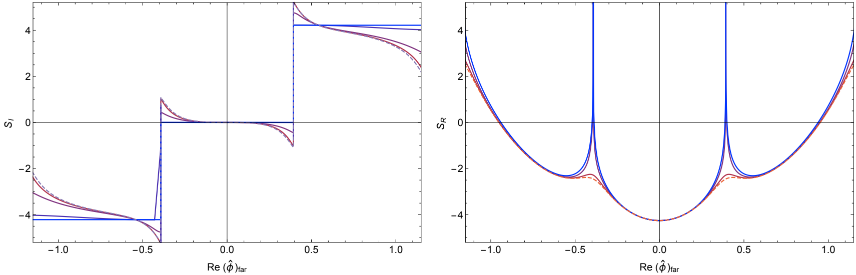
sign problem:

- complexification of fields is useful
- gradient flow ameliorates the sign problem
- introduces potential barriers, but can be managed
- finite density, real time ✓
- Future: better proposals, tempered transitions ?, estimators



model and data: 1510.03258, 1512.08764

tradeoff between sign problem and potential barriers:



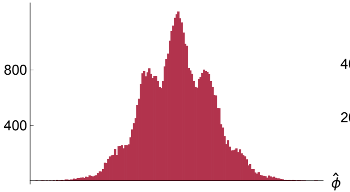
S_I S_R

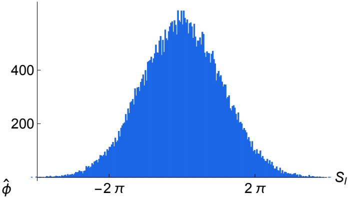
flow

sign problem $\Leftrightarrow$ multimodal distribution

model and data: 1510.03258, 1512.08764

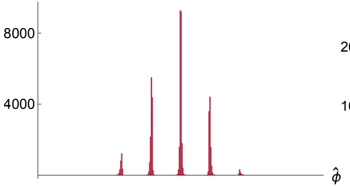
$T_{flow} = 0$

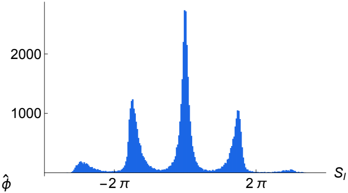




bad sign problem

$T_{flow} = 0.5$





multimodal distribution

mild sign problem

$$\text{sign problem} \quad \underset{\text{flow}}{\Leftrightarrow} \quad \text{multimodal distribution}$$

model and data: 1510.03258, 1512.08764